U-5,Two Degree of Freedom Vibration System

- * The vibrating systems, which require two coordinates to describe its motion, are called two-degrees-of -freedom systems.
- * These coordinates are called generalized coordinates when they are independent of each other and equal in number to the degrees of freedom of the system.
- * Unlike single degree of freedom system, where only one co-ordinate and hence one equation of motion is required to express the vibration of the system, in two-dof systems minimum two co-ordinates and hence two equations of motion are required to represent the motion of the system. For a conservative natural system, these equations can be written by using mass and stiffness matrices.
- * One may find a number of generalized co-ordinate systems to represent the motion of the same system. While using these co-ordinates the mass and stiffness matrices may be coupled or uncoupled. When the mass matrix is coupled, the system is said to be dynamically coupled and when the stiffness matrix is coupled, the system is known to be statically coupled.

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- * The set of co-ordinates for which both the mass and stiffness matrix are uncoupled, are known as principal co-ordinates. In this case both the system equations are independent and individually they can be solved as that of a single-dof system.
- * A two-dof system differs from the single dof system in that it has two natural frequencies, and for each of the natural frequencies there corresponds a natural state of vibration with a displacement configuration known as the normal mode. Mathematical terms associated with these quantities are eigenvalues and eigenvectors.
- * Normal mode vibrations are free vibrations that depend only on the mass and stiffness of the system and how they are distributed. A normal mode oscillation is defined as one in which each mass of the system undergoes harmonic motion of same frequency and passes the equilibrium position simultaneously.
- * The study of two-dof- systems is important because one may extend the same concepts used in these cases to more than 2-dof- systems. Also in these cases one can easily obtain an analytical or closed-form solutions. But for more degrees of freedom systems numerical analysis using computer is required to find natural frequencies (eigenvalues) and mode shapes (eigenvectors).

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- Figure 6.1.1 shows two masses m1 and m2 with three springs having spring stiffness k1, k2 and k3 free to move on the horizontal surface. Let x1 and x2 be the displacement of mass respectively.

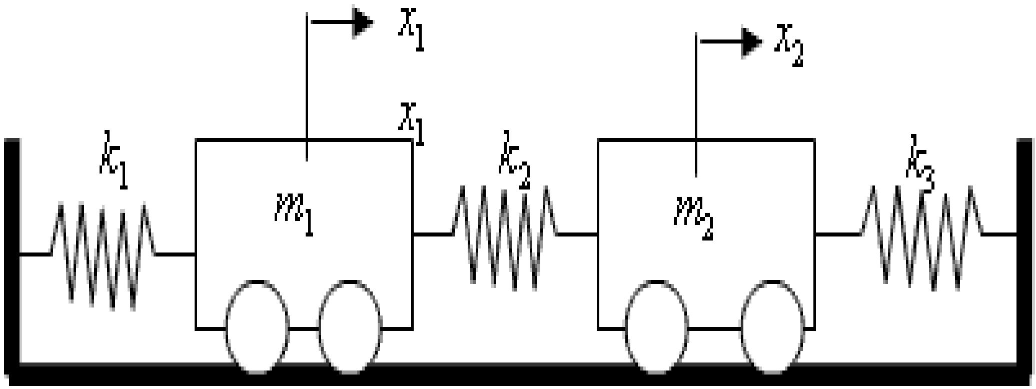

Figure 6.1.1(a)

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- by using d'Alembert principle or the energy principle (Lagrange principle or Hamilton 's principle)

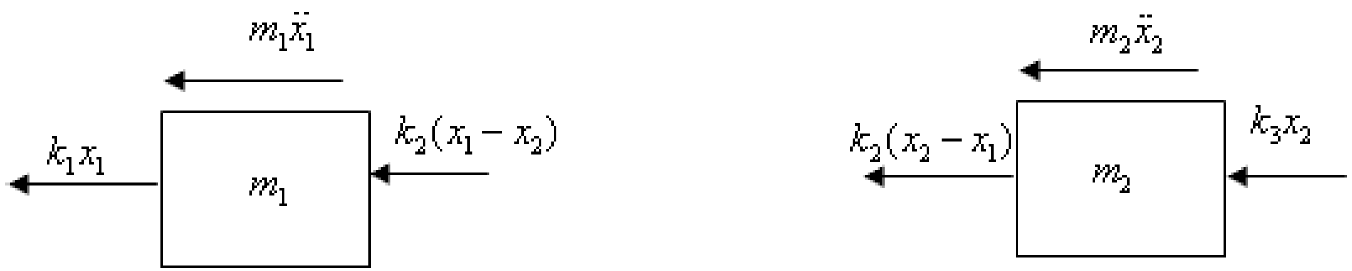

Figure 6.1.1(b): Free body diagrams

- Using d'Alembert principle for mass *m*1 from the free body diagram shown in figure 6.1.1(b)
- $$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \quad (6.1.1)$$
- and similarly for mass *m2*
- $$m_2\ddot{x}_2 - k_1 x_1 + (k_2 + k_3)x_2 = 0 \quad (6.1.2)$$

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Important points to remember

- Inertia force acts opposite to the direction of acceleration, so in both the free body diagrams inertia forces are shown towards left.

- For spring $m2$ assuming $x1 > x2$, The spring will pull mass $m2$ towards right by $k2\ (x2 - x1)$ and it is stretched by $x2 - x1$ (towards right) it will exert a force of $k2\ (x2 - x1)$ towards left on mass $m2$.
- Similarly assuming $x1 > x2$ the spring get compressed by an amount $x2 - x1$ and exert tensile force of $k2\ (x2 - x1)$. One may note that in both cases, free body diagram remain unchanged.

- Now if one uses Lagrange principle,

- The Kinetic energy = $T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$ (6.1.3)
- Potential energy = $U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_1 - x_2)^2 + \frac{1}{2} k_3 x_2^2$ (6.1.4)

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- So, the Lagrangian
- $$L = T - U = \left(\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2\right) - \left(\frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3x_2^2\right) \quad (6.1.5)$$
- The equation of motion for this free vibration case can be found from the Lagrange principle
- $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad (6.1.6)$$
- and noting that the generalized co-ordinate $q_1 = x_1$ and $q_2 = x_2$
- which yields
- $$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \quad (6.1.7)$$
-
- $$m_2\ddot{x}_2 - k_1x_1 + (k_2 + k_3)x_2 = 0 \quad (6.1.8)$$

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- Same as obtained before using d'Alembert principle.

- Now writing the equation of motion in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (6.1.9)$$

- Here it may be noted that for the present two degree-of-freedom system, the system is dynamically uncoupled but statically coupled.

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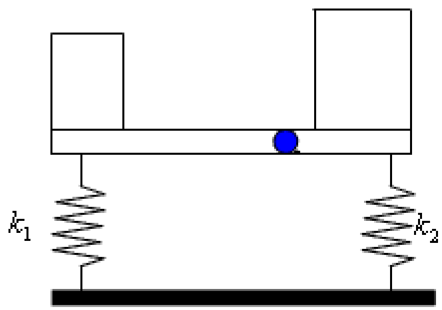
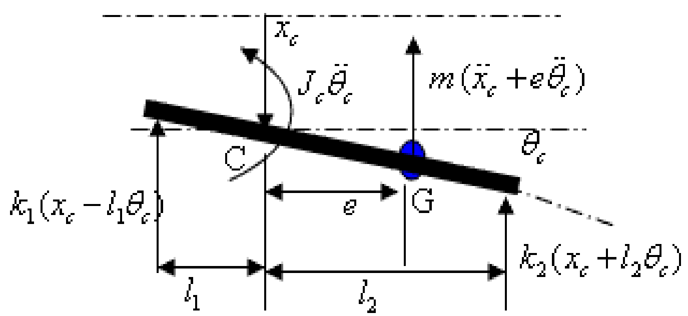
- Consider a lathe machine, which can be modeled as a rigid bar with its center of mass not coinciding with its geometric center and supported by two springs,
- 



- Figure 6.1.2 Figure 6.1.3: Free body diagram of the system
- In this example, it will be shown, how the use of different coordinate systems lead to static and or dynamic coupled or uncoupled equations of motion.

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- Clearly this is a two-degree-of freedom system and one may express the co-ordinate system in many different ways. Figure 6.1.3 shows the free body diagram of the system where point G is the center of mass. Point C represents a point on the bar at which we want to define the co-ordinates of this system. This point is at a distance from the left end and from right end. Distance between points C and G is e. Assuming is the linear displacement of point C and the rotation about point C, the equation of motion of this system can be obtained by using d'Alembert's principle. Now summation of all the forces, viz. the spring forces and the inertia forces must be equal to zero leads to the following equation.
- $$m\ddot{x}_c + me\ddot{\theta}_c + k_1(x_c - l_1\theta_c) + k_2(x_c + l_2\theta_c) = 0 \qquad (6.1.10)$$
- Again taking moment of all the forces about point C
- $$J_G\ddot{\theta}_c + (m\ddot{x}_c + me\ddot{\theta}_c)e - k_1(x_c - l_1\theta_c)l_1 + k_2(x_c + l_2\theta_c)l_2 = 0 \qquad (6.1.11)$$

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- Noting $J_c = J_G + me^2$, the above two equations in matrix form can be written as
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- $$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix} \begin{pmatrix} \ddot{x}_c \\ \ddot{\theta}_c \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{pmatrix} x_c \\ \theta_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (6.1.12)$$
- Now depending on the position of point C, few cases can be studied below.
- <u>Case 1</u> : Considering $e = 0$,i.e., point C and G coincides, the equation of motion can be written as
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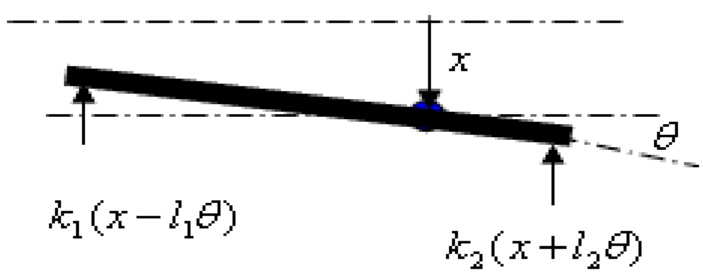

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- Figure 6.1.4

Two Degree of Freedom Vibration System

- Derivation of Equation of Motion and Coordinate Coupling
- $$\begin{bmatrix} m & 0 \\ 0 & J_G \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (6.1.13)$$
- So in this case the system is statically coupled and if $k_2 l_2 = k_1 l_1$, this coupling disappears, and we obtain uncoupled x and θ vibrations.
- <u>Case 2</u> : If, $k_1 l_1 = k_2 l_2$, the equation of motion becomes
- $$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix} \begin{pmatrix} \ddot{x}_c \\ \ddot{\theta}_c \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & 0 \\ 0 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{pmatrix} x_c \\ \theta_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (6.1.14)$$
- Hence in this ... d but statically uncoupled.

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- <u>Case 3</u>: If we choose $l_1 = 0$. , i.e. point C coincide with the left end,
- the equation of motion will become

$$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix} \begin{pmatrix} \ddot{x}_c \\ \ddot{\theta}_c \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 \\ k_2 l_2 & k_2 l_2^2 \end{bmatrix} \begin{pmatrix} x_c \\ \theta_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (6.1.15)$$

- Here the system is both statically and dynamically coupled.

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- **Normal Mode Vibration**
- Again considering the problem of the spring-mass system in
- figure 6.1.1 with $m_1 = m$. $m_2 = 2m$. $k_1 = k_2 = k_3 = k$., the equation of motion (6.1.9) can be written as
- $$m\ddot{x}_1 + k(x_1 - x_2) + kx_1 = 0$$
- $$2m\ddot{x}_2 - k(x_1 - x_2) + kx_2 = 0 \qquad (6.1.16)$$
- We define a normal mode oscillation as one in which each mass undergoes harmonic motion of the same frequency, passing simultaneously through the equilibrium position. For such motion, we let
- $$x_1 = A_1 e^{i\omega t}, x_2 = A_2 e^{i\omega t} \qquad (6.1.17)$$
- Hence,
- $$(2k - m\omega^2)A_1 - kA_2 = 0$$
- $$-kA_1 + (2k - 2m\omega^2)A_2 = 0 \qquad (6.1.18)$$

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- or, in matrix form

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - 2m\omega^2 \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (6.1.19)$$

- Hence for nonzero values of A_1 $\lambda^2 - (3\frac{k}{m})\lambda + \frac{3}{2}(\frac{k}{m})^2 = 0$ for non-trivial response)

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - 2m\omega^2 \end{vmatrix} = 0 \qquad (6.1.20)$$

- Now substituting $\omega^2 = \lambda$, equation 6.1. yields

$$\lambda^2 - (3\frac{k}{m})\lambda + \frac{3}{2}(\frac{k}{m})^2 = 0 \qquad (6.1.21)$$

- Hence, $\lambda_1 = (\frac{3}{2} - \frac{1}{2}\sqrt{3})\frac{k}{m} = 0.634\frac{k}{m}$ and $\lambda_2 = (\frac{3}{2} + \frac{1}{2}\sqrt{3})\frac{k}{m} = 2.366\frac{k}{m}$

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- So, the natural frequecies of the system are $\omega_1 = \sqrt{\lambda_1} = \sqrt{0.634\frac{k}{m}}$ and
- $$\omega_2 = \sqrt{2.366\frac{k}{m}}$$
- Now from equation 6.1., it may be observed that for these frequencies, as both the equations are not independent, one can not get unique value of and . So one should find a normalized value. One may normalize the response by finding the ratio of to . From the first equation 6.1. the normalized value can be given by
- $$\frac{A_1}{A_2} = \frac{k}{2k - m\omega^2} = \frac{k}{2k - m\lambda} \qquad (6.1.22)$$
- and from the second equation of 6.1., the normalized value can be given by
- $$\frac{A_1}{A_2} = \frac{2k - 2m\omega^2}{k} = \frac{2k - 2m\lambda}{k} \qquad (6.1.23)$$

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- Now, substituting $\omega_1^2 = \lambda_1 = 0.634\frac{k}{m}$ in equation 6.1.22 and 6.1.23 yields the same values, as both these equations are linearly dependent. Here,

$$\left(\frac{A_1}{A_2}\right)_{\lambda=\lambda_1} = \frac{0.732}{1} \quad (6.1.24)$$

- and similarly for $\omega_2^2 = \lambda_2 = 2.366\frac{k}{m}$

$$\left(\frac{A_1}{A_2}\right)_{\lambda=\lambda_2} = \frac{-2.73}{1} \quad (6.1.25)$$

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It may be noted

- Equation (6.1.19) gives only the ratio of the amplitudes and not their absolute values, which are arbitrary.
- If one of the amplitudes is chosen to be 1 or any number, we say that amplitudes ratio is normalized to that number.
- The normalized amplitude ratios are called the normal modes and designated by $\phi_n(x)$.
- From equation 6.1.24 and 6.1.25, the two normal modes of this problem are:
- $$\phi_1(x) = \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \qquad \phi_2(x) = \begin{Bmatrix} -2.73 \\ 1.00 \end{Bmatrix}$$
- In the 1st normal mode, the two masses move in the same direction and are said to be in phase and in the 2nd mode the two masses move in the opposite direction and are said to be out of phase. Also in the first mode when the second mass moves unit distance, the first mass moves 0.731 units in the same direction and in the second mode, when the second mass moves unit distance; the first mass moves 2.73 units in opposite direction.

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- When the system is disturbed from its initial position, the resulting free-vibration of the system will be a combination of the different normal modes. The participation of different modes will depend on the initial conditions of displacements and velocities. So for a system the free vibration can be given by
- $$x = \phi_1 A \sin(\omega_1 t + \psi_1) + \phi_2 B \sin(\omega_2 t + \psi_2)$$
- Here *A* and *B* are part of participation of first and second modes respectively in the resulting free vibration and ψ_1 nd ψ_2 are the

  phase difference. They depend on the initial conditions. This is explained with the help of the following example.
- **Example** :
- Let us consider the same spring-mass problem (figure 6.2.1) for which the natural frequencies and normal modes are determined. We have to determine the resulting free vibration when the system is given an initial displacement

Two Degree of Freedom Vibration System

- Free vibration using normal modes

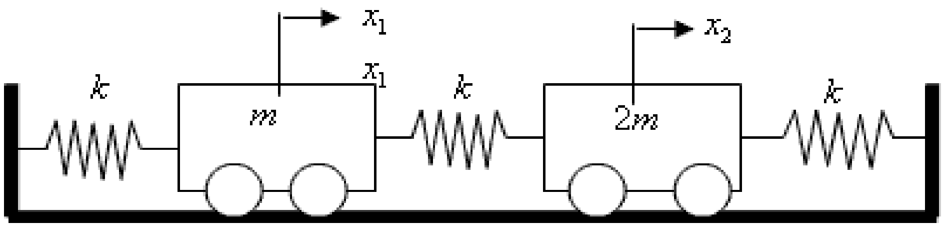

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Figure 6.2.1

- Solution **:** Any free vibration can be considered to be the superposition of its normal modes. For each of these modes the time solution can be expressed as:

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$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \sin \omega_1 t$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 = \begin{Bmatrix} -2.731 \\ 1.00 \end{Bmatrix} \sin \omega_2 t$$

Two Degree of Freedom Vibration System

- The general solution for the free vibration can then be written as:
- $$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = A \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \sin(\omega_1 t + \psi_1) + B \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \sin(\omega_2 t + \psi_2) .$$
- where *A* and *B* allow different amounts of each mode and ψ_1
- ψ_2 and allows the two modes different phases or starting values.
- Substituting:
- $$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 5 \\ 1 \end{Bmatrix} = A \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \sin\psi_1 + B \begin{Bmatrix} -2.731 \\ 1 \end{Bmatrix} \sin\psi_2$$
- $$\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \omega_1 A \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \cos\psi_1 + \omega_2 B \begin{Bmatrix} -2.731 \\ 1 \end{Bmatrix} \cos\psi_2 .$$

$$\cos\psi_1 = \cos\psi_2 = 0 \Rightarrow \psi_1 = \psi_2 = 90^0$$

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- Substituting in 1st set:

$$\begin{Bmatrix} 5 \\ 1 \end{Bmatrix} = A\begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} + B\begin{Bmatrix} -2.731 \\ 1 \end{Bmatrix}.$$

$$\left.\begin{aligned} 0.731A - 2.731B &= 5 \\ A + B &= 1 \end{aligned}\right\} \begin{aligned} A &= 2.233 \\ B &= -1.233 \end{aligned}$$

- Hence the resulting free vibration is

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 2.233\begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix}\cos\omega_1 t - 1.233\begin{Bmatrix} -2.731 \\ 1.000 \end{Bmatrix}\cos\omega_2 t.$$

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- Normal modes from eigenvalues
- The equation of motion for a two-degree-of freedom system can be written in matrix form as

$$M\ddot{x} + Kx = 0 \qquad (6.2.)$$

- where M and K are the mass and stiffness matrix respectively; x is is the vector of generalized co-ordinates. Now pre-multiplying in both side of equation 6.2. one may get

$$I\ddot{x} + M^{-1}Kx = 0$$

or,

$$I\ddot{x} + Ax = 0$$

- where $A = M^{-1}K$ is known as the dynamic matrix. Now to find the normal modes,
- $x_1 = X_1 e^{i\omega t}, x_2 = X_2 e^{i\omega t}$, the above equation will reduce to
- $[A - \lambda I]X = 0$ where $X = \{x_1 \quad x_2\}^T$ and $\lambda = \omega^2$

From equation 6.2. it is apparent that the free vibration problem in this case is reduced to that of finding the eigenvalues and eigenvectors of the matrix A .